BVG Trust

J. Sci. Trans. Environ. Technov. 2012, 6(2) : 91 - 94

Scientific Transactions in Environment and Technovation

Fundamental theorem of homomorphism in fuzzy subgroups

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Abstract

In this paper, a new concept of homomorphism between two fuzzy subgroups μ and μ′is defined. Many results analogous to homomorphism of groups are established and Fundamental theorem of homomorphism in fuzzy subgroups is derived.

Keywords: fuzzy homomorphism, fuzzy isomorphism, fuzzy set, kernel.

INTRODUCTION

After the introduction of fuzzy sets by Zadeh. Zimmerman (2001), several reachers explored on the generalization of the notion of fuzzy set. Vasantha Kandasamy (2003) defined fuzzy subgroups. We define the concepts of homomorphism in fuzzy subgroups and fuzzy homomorphism, fuzzy isomorphism, Kernel in fuzzy, onto , one to one. We derive fundamental theorem of homomorphism in fuzzy subgroups and establish some results in this paper.

1. Basics

Definition: 1.1

Let x be a non empty set. A fuzzy set μ of the set

x is a function $\mu : x \rightarrow [0,1]$.

Definition: 1.2

The most commonly used range of membership functions is the unit Interval [0, 1]. In this case, each membership function maps elements of a given universal set x, which is always a crisp set, into real numbers in [0,1]. Two distinct notations are most commonly employed in the literature to denote membership functions.

The membership functions of a fuzzy set A is denoted by

$$\mu \quad \tilde{A} : x \rightarrow [0, 1]$$

Definition: 1.3

Let G be a group. A fuzzy subset μ of a group G is called a fuzzy Subgroup of the group G if.

i. $\mu(xy) \geq \min\{\mu(y)\}$ for every $x, y \in G$ and

ii. $\mu(x^{-1}) = \mu(x)$ for every $x \in G$

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Definition: 1.4

Let μ be a fuzzy subgroup of a group G and $H = \{x \in G / \mu(x) = \mu(e)\}$ then, order of μ is defined as $o(\mu) = o(H)$

Definition: 1.5

Let G be a group. A fuzzy subgroup A of G is called normal if

$A(x) = A(y^{-1}xy)$ for all $x, y \in G$.

Definition: 1.6

Let A be a fuzzy subset of S for $t \in [0,1]$ the set $A_t = \{ s \in S / A(x) \geq t \}$ is called s level Subset of the fuzzy subset A.

Definition: 1.7

A fuzzy subgroup μ at a group G is called improper if μ is constant of the group G. Otherwise μ is termed as proper.

Definition: 1.8

Let μ be a fuzzy subgroup of a group G for any $a \in G$, aμ defined by $(a\mu)x = \mu(a^{-1}x)$ For every $x \in G$ is called the fuzzy coset of the group G determined by a and μ.

Definition: 1.9

Let λ and μ be two fuzzy subgroups of a group G. Then λ and μ are said to be Conjugate fuzzy subgroups of G if for

some $g \in G$, $\lambda(x) = \mu(g^{-1}xg)$ for every $x \in G$.

Definition: 1.10

Let μ and μ′ be any two fuzzy subsets. The mapping $f : \mu \rightarrow \mu'$ is onto , every Element of μ′ has a pre - image in μ. $f(\mu_x) = \mu_{x'}$

Definition: 1.11

A function f : μ′ is one – one if distinct elements in A have distinct images under f.

$f(\mu_{x1}) = f(\mu_{x2})$,

$\mu_{x1} = \mu_{x2}$

Definition: 1.12
A map from a fuzzy subgroup μ into a fuzzy subgroup μ′ is called a fuzzy Homomorphism if f $(\mu_x\ \mu_y) = f(\mu_x)\, f(\mu_y)$

Definition: 1.13
Let f : μ → μ′ be a homomorphism if f is onto, then it is called an epimorphism.

Definition: 1.14
Let μ and μ′ be two fuzzy subgroups A map f : μ → μ2 is called a fuzzy isomorphism If

(i) f is a bijection

(ii) $f(\mu_x \mu_y) = f(\mu_x)\, f(\mu_y)$

Definition: 1.15
Let f : μ → μ′ be a homomorphism.

Let $K = \{ \mu_x / \mu_x \varepsilon \mu, f(\mu_x) = \mu_{e'} \}$, then K is called The Kernel of f and it denoted by kerf.

2. Properties of Fuzzy subgroup :

Theorem : 2.1
A Fuzzy subset μ of a group G is a fuzzy subgroup of the group G if and only if

$\mu(xy^{-1}) \geq \min\{\mu(y)\}$ for every x, yεG.

Proof : Let G be a group

Let μ be a fuzzy subgroup of a group G, prove that
$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$

Since $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

=>Replace y by y^{-1}, y ε G => y^{-1} ε G

$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(y) = \mu(y^{-1})$
=>$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$

Conversely,

Let $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$
Prove that μ is a Fuzzy sub group of group G.
$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$
=> Replace y^{-1} by y, y^{-1}ε G => y ε G

=>$\mu(x,y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(y) = \mu(y^{-1})$
μ is a fuzzy subgroup of group G.
Hence the theorem.

Theorem : 2.2
Let μ be a fuzzy subgroup of a group G and x ε G.
Then μ (xy) = μ (y) for every y ε G. if and only if μ (x) = μ(e).

Proof :

Let G be a group and μ be a fuzzy subgroup of group G

Let μ (xy) = μ (y), Prove that μ (x) = μ (e)
Since μ is a subgroup of group G.

$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ x, y ε G. and μ (x) = μ (x) , x ε G

Let e be an identity element of group G.

$\mu(xy) \geq \min\{\mu(x), \mu(y)\} \to 1$

=>Replace x by e

$\mu(ey) \geq \min\{\mu(e), \mu(y)\}$

$\mu(y) \geq \min\{\mu(e), \mu(y)\} \to 2$

Since 'e' identity element.

(Since μ (xy) = μ (y))

μ (x) = μ (e)

Conversely,

Let μ (x) = μ (e), Prove that μ (xy) = μ (y)

=> $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

=> $\mu(y) \geq \min\{\mu(e), \mu(y)\}$

Since μ (x) = μ (e)

So, 2 => $\mu(y) \geq \min\{\mu(x), \mu(y)\}$

=>So, μ (xy) = μ (y)

μ (xy) = μ (y) Hence the theorem.

Theorem : 2.3

Show that the intersection of two fuzzy normal subgroups of G is a fuzzy normal subgroup of G.

Proof :
Suppose G is a group.
Let μ and λ be two fuzzy normal subgroups of G.
Let μ be a fuzzy normal subgroups of G.
=> $\mu(x) = \mu(y^{-1}xy)$; x, y ε G

and λ be a fuzzy normal fuzzy subgroups of G.

=> $\lambda(X) = \lambda(y^{-1}xy)$, x, y, ε G
=> $\mu(x) \cap \lambda(x) = \mu(y^{-1}xy) \cap \lambda(y^{-1}xy)$, x, y,ε G
=> $(\mu \cap \lambda)(x) = (\mu \cap \lambda)(y^{-1}xy)$, x, y, ε G

μ∩λ is fuzzy normal subgroup of G.
Hence the theorem.

3. FUNDAMENTAL THEOREMOF HOMOMORPHISM IN FUZZY SUBGROUP

Theorem : 3.1

If G is a fuzzy subgroup and μ is a normal fuzzy subgroup of G then G/ μ is also a fuzzy subgroup under the product of subsets of G.

Proof : Let $(a\mu)(x) = \mu(a^{-1}x)$

lll^{ly} $(b\mu)(x) = \mu(b^{-1}x)$, $a\mu, b\mu \varepsilon G$

$(ab\mu)x = \mu(ab)^{-1}x = \mu(b^{-1}a^{-1})x$

(i) $(a\mu)(x)(b\mu)(x) = \mu(a^{-1}x)\mu b^{-1}x$
$= \mu(a^{-1}b^{-1})x$
$= \mu(b^{-1}a^{-1})x$

$= \mu(b^{-1}a^{-1})x$
$= \mu((ab)^{-1})x$
$= (ab)\mu(x)$

G/ μ is closed

(ii) $[(a\mu(x)b\mu(x)]c\mu(x)] = [\mu(a^{-1}x)\mu(b^{-1}x)]\mu(c^{-1}x)$

$= [\mu(a^{-1}b^{-1}x]\mu(c^{-1}x)$

$= \mu(a^{-1}b^{-1}c^{-1}x)$

$= (a\mu)x(bc\mu)x$

$= (a\mu)x[b\mu(x)c\mu(x)]$

G/μ is associative

(iii) $(e\mu)(x)(a\mu)(x) = \mu(e^{-1}x)\mu(a^{-1}x)$
$= \mu(e^{-1}a^{-1}x)$, e, e^{-1} åG
$= \mu(a^{-1}x)$
$= (a\mu)\mu(x)$

lll^{ly} $(a\mu)x\ e\mu(x) = (a\mu)x$

G/μ is satisfied identity

(iv) $(a\mu)(x)(a^{-1}\mu)(x) = \mu(a^{-1}x)\mu[(a^{-1})^{-1})x]$
$= \mu(a^{-1}x)\mu(ax)$ $(a^{-1})^{-1}=a$

$= \mu(a^{-1}ax)$

$= \mu(x)$

$(a^{-1}\mu)x(a\mu)x = \mu(x) = (a\mu)x(a^{-1}\mu)x$

=> G/μ is a Inverse.

=> The fuzzy coset aμ satisfies all the group axioms
=> G/μ = { aμ /aεG } is a group.
Now

$G/\mu\ (a\mu b\mu) = G/\mu\ (ab\mu) \geq \min[G/\mu(a\mu), G/\mu(b\mu)]$

For every aμ, bμε G, $G/\mu[(a\mu)^{-1}] = G/\mu(a\mu)$

=> G/μ is a fuzzy subgroup
Hence the Theorem.

Definition: 3.1
If G is a fuzzy subgroup and μ is a normal fuzzy subgroup of G then G/μ is also a fuzzy subgroup under the product of subsets of G. It is called fuzzy quotient subgroup or fuzzy factor subgroup of G by μ.

Theorem : 3.2

Fundamental theorem of homomorphism in fuzzy subgroup :

Let μ and μ_2 be a two fuzzy subgroups,, if f : μ '! μ_2 be an epimophism Let K be the Kernel of f. Then μ/K isomorphic to μ_2.

Proof :

Define $\phi: \mu/k \to \mu'$

$f: \mu \to \mu'$ => for all $\mu_x \varepsilon\ \mu, \mu_{x'} \varepsilon\ \mu'$ => $f(\mu_x) = \mu_{x'}$

μ_e is the identity element of μ

$\phi: \mu/k \to \mu'$ => for all $k\mu_e f \mu/k$ => $\phi(k\mu_x) = f(\mu_x)$

Step : 1

φ is well defined

Let $k\mu_x = k\lambda_x$, for all $\mu_x, \lambda_x \varepsilon \mu$

then $\lambda_x \varepsilon k\mu_x$

$\lambda_x = k\mu_x . k \varepsilon k$

Now , $f(\lambda_x) = f(k\mu_x) = f(k) f(\mu_x)$
$= \mu_{e'} f(\mu_x)$
$= f(\mu_x)$

therefore $\phi(k\lambda_x) = f(\lambda_x)$

$= f(\mu_x)$

$\phi(k\lambda_x) = \phi(k\mu_x)$

Step : 2

φ is one - one

for, $\phi(k\mu_x) = \phi(k\lambda_x)$

$f(\mu_x) = f(\lambda_x)$

$f(\mu_x)[f(\lambda_x)]^{-1} = \mu_{e'}$

$\mu_x(\lambda_x)^{-1} \varepsilon k$

$\mu_x \varepsilon k\lambda_x$

$=> k\mu_x = k\lambda_x$

Step : 3

φ is onto

Let $\mu_{x2} \varepsilon \mu'$

Since f is onto there exists μ_{x2} ε μ Such that $f(\mu_x) = \mu_{x'}$

Hence $\phi(k\mu_x) = f(\mu_x) = \mu_{x2}$

Step : 4

φ is a homomorphism

$\phi(k\mu_x k\lambda_x) = \phi(k\mu_x \lambda_x)$

$= f(\mu_x \lambda_x)$

$= f(\mu_x)\, f(\lambda_x)$ (f is homomorphism)

$= \phi(k\mu_x)\, \phi(k\lambda_x)$

Thus ϕ is isomorphism from μ/k onto μ'

Conclusion

In this paper, with the basic concepts of group theory, we define related concepts of fuzzy group theory and further we define a fuzzy isomorphism in fuzzy subgroups and hence conclude fundamental theorem of homomorphism in fuzzy subgroups.

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