BVG Trust

J. Sci. Trans. Environ. Technov. 2010, 4(1): 37-38

Scientific Transactions in Environment and Technovation

Parametric Representation of Indices of $O_a + O_c = 2O_b$

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Abstract

In this paper, an attempt has been made to present the observation on octagonal numbers in arithmetic progression.

Keywords: arithmetic progression, indices, octagonal numbers, parametric representation, Pell equation, Pythagorean triangle and triplet.

I. INTRODUCTION

The polygonal numbers are positive integers that can be represented by regular polygons in a systematic fashion. There are various types of such numbers: triangular numbers, square numbers, pentagonal numbers, hexagonal numbers, heptagonal numbers and octagonal numbers.

Here we deal with the octagonal numbers in arithmetic progression. The octagonal number is a positive integer that can be represented in octagonal array. The n^{th} octagonal number is denoted by O_n, $n \geq 1$. The octagonal numbers form the sequence : 1; 8; 21; 40; 65; 96; 113; 176; ………… and the general formula [1,2] is $O_r = r(3r-2)$.
Three octagonal numbers may be in arithmetic progression and the proof of the four or more numbers are difficult.

In this paper, it has been observed that there exists a triple of octagonal numbers in arithmetic progression and that there exists infinitely many triples of octagonal numbers in arithmetic progression with identical least value by using the theory of Pell's equation.

II. EXISTENCE OF TRIPLETS OF THE OCTAGONAL NUMBERS IN ARITHEMETIC PROGRESSION.

Method of Analysis: Let the three octagonal numbers O_a, O_b, O_c be in arithmetic progression, we have

$$O_a + O_c = 2O_b \quad \text{.......(2.1)}$$

Here we would like to prove that two parametric representations of indices of the three octagonal numbers O_a,O_b and O_c are satisfying equation(2.1). The equation (2.1) is written as

$$X^2 + Z^2 = 2Y^2 \quad \text{..........(2.2)}$$

where

$$X = (3a-1)/2\ ,\ Y = (3b-1)/2 \text{ and } Z = (3c-1)\ /2 \quad \text{......(2.3)}$$

If

$$X = p - q \text{ and } Z = p + q \quad \text{......(2.4)}$$

Then

$$X^2 + Z^2 = 2(p^2 + q^2) \quad \text{......(2.5)}$$

By comparing the equation (2.2) with equation(2.5),we get $Y^2 = p^2 + q^2$.

It is famous Pythagorian triangle equation. Its solution is given by

$$p = m^2 - n^2\ ,\ q = 2mn \text{ and } Y = m^2 + n^2 \quad \text{.......(2.6)}$$

The value of m and n are distinct positive integers. They are arbitrarily selected. From (2.4) and (2.6), we have

$$X = m^2 - n^2 - 2mn,$$
$$Y = m^2 + n^2,$$
$$Z = m^2 - n^2 + 2mn$$

Thus, from the equation (2.3),we have

$$a = (2(m^2-n^2-2mn)+ 1)\ /\ 3,$$
$$b = (2(m^2 + n^2)+ 1)\ /3, \quad \text{..........(2.7)}$$
$$c = (2(m^2-n^2 + 2mn)+1)\ /\ 3.$$

It is possible to choose m and n suitably so that a , b and c are integers.

Thus, the integer values of a,b and c (from (2.7)) represent the ranks of the octagonal numbers O_a,O_b,O_c and they satisfy the equation (2.1).

Therefore, it is clear to note that there exists triplets of octagonal numbers in arithmetic progression.

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INFINITELY MANY TRIPLES OF OCTAGONAL NUMBERS IN ARITHMETIC PROGRESSION WITH IDENTICAL LEAST VALUES.

Method of Analysis: Here we would like to prove that there exist infinitely many triples of octagonal numbers in arithmetic progression with identical least value by using the theory of Pell's equation. We remind the following formula (from (2.7)).

$$a = (2(m^2 - n^2 - 2mn) + 1) / 3$$
$$b = (2(m^2 + n^2) + 1) / 3$$
$$c = (2(m^2 - n^2 + 2mn) + 1) / 3$$

where a,b and c are indices of the three octagonal numbers satisfying the equation $O_a + O_c = 2O_b$.
From equation (2.4) [x = p-q, z = p + q], the values of p and q should be chosen that $p - q \geq 1$. Then, the values of m and n (from(2.6)) satisfy

$$m^2 - n^2 - 2mn \geq 1.$$

It is written as

$$(m-n)^2 - 2n^2 \geq 1.$$

This leads to

$$(m-n)^2 - 2n^2 = m^2 - n^2 - 2mn \geq 1.$$

This is a Pell equation of the type,

$$U_t^2 - 2V_t^2 = m_t^2 - n_t^2 - 2m_t n_t \geq 1 \qquad \ldots\ldots.(3.1)$$

where $U_t = m_t - n_t$ and $V_t = n_t$ and hence, it has different classes of infinitely many solutions.

Let there be β classes of solution sequence of Pell equation (3.1). Let it be given by

$$U_{\alpha,t} + V_{\alpha,t}\sqrt{2} = (u_\alpha + v_\alpha \sqrt{2})(3+2\sqrt{2})^t \qquad \ldots\ldots(3.2)$$

where $u_\alpha + v_\alpha \sqrt{2}$ run through all the β fundamental solutions of $U_t^2 - 2v_t^2 > 1$. Using $U_{\alpha,t} = m_{\alpha,t} - n_{\alpha,t}$ and $V_{\alpha,t} = n_{\alpha,t}$ in (3.2), we get $m_{\alpha,t} = U_{\alpha,t} + V_{\alpha,t}$
That is,

$$m_{\alpha,t} = ((U_\alpha + V_\alpha\sqrt{2})(3+2\sqrt{2})^t(2+\sqrt{2}) + (u_\alpha - v_\alpha\sqrt{2})(3-2\sqrt{2})^t(2-\sqrt{2}))/4$$

and

$$n_{\alpha,t} = ((u_\alpha + v_\alpha\sqrt{2})(3+2\sqrt{2})^t - (u_\alpha - v_\alpha\sqrt{2})(3-2\sqrt{2})^t)/2\sqrt{2}$$

In view of equation (2.7),we have the positive integers of the form

$$a = (2(U_t^2 - 2V_t^2) + 1) / 3,$$
$$b = (2(U_t^2 + 2V_t^2 + 2U_tV_t) + 1) / 3, \qquad \ldots\ldots (3.3)$$
$$c = (2((U_t + 2V_t)^2 - 2V_t^2) + 1) / 3.$$

These positive integers a,b,c are the ranks of the octagonal numbers in arithmetic progression. It can be easily seen that many triples of octagonal numbers in arithmetic progression with identical least values.

For the sake of simplicity and investigation, we present a few positive integer values of U and V as follows:

(i)
$$U = ((3+2\sqrt{2})^{2(t-1)}(5+3\sqrt{2}) + (3-2\sqrt{2})^{2(t-1)}(5-3\sqrt{2}))/2$$
$$V = ((3+2\sqrt{2})^{2(t-1)}(5+3\sqrt{2}) - (3-2\sqrt{2})^{2(t-1)}(5-3\sqrt{2}))/2\sqrt{2}$$

(ii)
$$U = ((3+2\sqrt{2})^{2(t-1)}(7+3\sqrt{2}) + (3-2\sqrt{2})^{2(t-1)}(7-3\sqrt{2}))/2$$
$$V = ((3+2\sqrt{2})^{2(t-1)}(7+3\sqrt{2}) - (3-2\sqrt{2})^{2(t-1)}(7-3\sqrt{2}))/2\sqrt{2}$$

(iii)
$$U = ((3+2\sqrt{2})^{2(t-1)}(8+3\sqrt{2}) + (3-2\sqrt{2})^{2(t-1)}(8-3\sqrt{2}))/2$$
$$V = ((3+2\sqrt{2})^{2(t-1)}(8+3\sqrt{2}) - (3-2\sqrt{2})^{2(t-1)}(8-3\sqrt{2}))/2\sqrt{2}$$

(iv)
$$U = ((3+2\sqrt{2})^{2(t-1)}(10+3\sqrt{2}) + (3-2\sqrt{2})^{2(t-1)}(10-3\sqrt{2}))/2$$
$$V = ((3+2\sqrt{2})^{2(t-1)}(10+3\sqrt{2}) - (3-2\sqrt{2})^{2(t-1)}(10-3\sqrt{2}))/2\sqrt{2}$$

(v)
$$U = ((3+2\sqrt{2})^{2(t-1)}(11+3\sqrt{2}) + (3-2\sqrt{2})^{2(t-1)}(11-3\sqrt{2}))/2$$
$$V = ((3+2\sqrt{2})^{2(t-1)}(11+3\sqrt{2}) - (3-2\sqrt{2})^{2(t-1)}(11-3\sqrt{2}))/2\sqrt{2}$$

(vi)
$$U = ((3+2\sqrt{2})^{2(t-1)}(10+6\sqrt{2}) + (3-2\sqrt{2})^{2(t-1)}(10-6\sqrt{2}))/2$$
$$V = ((3+2\sqrt{2})^{2(t-1)}(10+6\sqrt{2})) - (3-2\sqrt{2})^{2(t-1)}(10-6\sqrt{2}))/2\sqrt{2}$$

(vii)
$$U = ((3+2\sqrt{2})^{2(t-1)}(17+12\sqrt{2}) + (3-2\sqrt{2})^{2(t-1)}(17-12\sqrt{2}))/2$$
$$V = ((3+2\sqrt{2})^{2(t-1)}(17+12\sqrt{2}) - (3-2\sqrt{2})^{2(t-1)}(17-12\sqrt{2}))/2\sqrt{2}$$

where t = 1,2,3,…………..
These values are satisfying the equation (3.3) to intergers

m	n	a	b	c	o_a	o_b	o_c	Common Difference
8	3	5	49	69	65	7105	14145	7040
10	3	21	73	101	1281	15841	30401	14560
11	3	31	87	119	2821	22533	42245	19712
13	3	55	119	159	8965	42245	75525	33280
14	3	69	137	181	14145	56033	97921	41888
16	6	19	195	275	1045	113685	226325	112640
29	12	1	657	929	1	1293633	2587265	1293632

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