The Geometry of the Virtual knots

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Abstract

The Dehn complex of prime, alternating virtual links has been shown to be non-positively curved earlier. This paper investigates the geometry of an arbitrary alternating virtual link. A method is constructed for which the Dehn complex of any alternating virtual link may be decomposed into Dehn complexes with non-positive curvature. We further study the relationship between the Dehn space and Wirtinger space, and we relate their fundamental groups using generating curves on surfaces. We conclude with interesting examples of Dehn complexes of virtual link diagrams, which illustrate our findings.

Key words: Dehn complex, Projection surface, Pseudo manifold, Squared complex, Virtual knots.

INTRODUCTION

In the paper "Generalized knot complements and some aspherical ribbon disc complements" by Harlander and Rosebrock (2003) .The topological and geometric aspects of classical knot complements have been intensely studied. First results related to this thesis include work by Aumann (1969), who used combinatorial topological techniques on the Dehn complex to show the asphericity of alternating knots, and work by Weinbaum (1971), who studied knot complements in terms of small cancellation theory. Modern treatments of the curvature of knot complements have been given by Wise (2001) in terms of non-positively curved squared complexes, and this viewpoint was developed by Bridson and Haefliger (1999).

This paper investigates the topological and geometric aspects of the Dehn complex of a virtual knot. We are particularly interested in when a virtual knot admits a Dehn complex that is a non-positively curved squared complex. There are various descriptions of the fundamental group of the Dehn complex. One is obtained by collapsing an edge in the Dehn complex, and the other is obtained by coning off a surface in the Wirtinger complex. All these descriptions can be read off the virtual knot drawn on its projection surface.

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Theorem1:

Bridson/Haefliger (2017)

If $K \subset R3$ is an alternating link then $\pi 1(R3- K)$ is the fundamental group of a compact 2-dimensional piecewise-Euclidean 2-complex of non-positive curvature.

Harlander and Rosebrock (2003) extended this result to prime alternating virtual links. They used a strong version of primeness.

THE VIRTUAL KNOT AND VARIOUS COMPLEMENTS

Virtual knots and links are an extension of classical knot theory and were first introduced by Kauffman and Manturov (2006).

Definition 1

Virtual Link Diagrams

A virtual link diagram *l* is a 4-regular graph in the plane, with over-crossing and under-crossing information at some nodes. The nodes with this additional information are crossings, and the remaining nodes are called virtual crossings. The latter are indicated by a circle around the node. From a virtual link diagram in the plane, we construct a closed, orientable surface.

Definition 2

VARIOUS VIRTUAL LINK COMPLEMENTS

Given a virtual link diagram l,

Let F be a projection surface. We can push the diagram into the interior of the thickened surface F × I, and thus obtain an embedding of circles in F × I. If we

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remove an open neighborhood of from F ×I, then we obtain a compact manifold with boundary, which we will call the manifold space, M(l). If we cone off the bottom surface F ×{0} of M(l), we obtain a compact pseudomanifold with boundary, which we call the Wirtinger space, W(l). It turns out that W(l) can be collapsed to the 2-dimensional Wirtinger complex, W(l). If we cone off the top surface in the Wirtinger space, we obtain another compact pseudomanifold with boundary, which we call the Dehn space, D(l). The Dehn space can be collapsed down to the 2-dimensional Dehn complex, D(l). See Harlander and Rosebrock (1960) for details on this collapsing procedure on the Wirtinger space and the Dehn space.

NON-POSITIVE CURVATURE OF THE DEHN COMPLEX

Non-Positive Curvature:

A squared 2-complex is a complex where every 2-cell has four edges. The Dehn 2-complex and the Wirtinger2-complex are examples of a squared 2-complex.

Non-Positive Curvature.

A squared 2-complex is non-positively curved if there are at least four squares grouped around each vertex. That is, if every reduced edge cycle in the link graph of each vertex in the 2-complex has length at least four.

[Harlander and Rosebrock (2003)].

Theorem 2:

Harlander/Rosebrock (2003)

A virtual link diagram l is prime, alternating if and only if the Dehn complex of l is a non-postively curved squared complex.

Proof:

Suppose that l is alternating with a projection surface F.Since all edge cycles in Lk(v±, D(l)) are of even length. We have a 2-cycle in Lk(v±, D(l)) only if a pair of connected components of F "l, (Ai , Aj), about two distinct edges in l.

Definition 3:

We triple a virtual link diagram l by adding additional links, which run parallel on either side of all the original strands of the link. Denote the tripled link by l_T .

Theorem 3:

If l_T is an alternating triple of any alternating virtual link diagram l, then $D(l_T)$ is a non-positively curved squared complex.

Proof.

Let l_T be the triple of l that is alternating. If we can show l_T is prime, it implies $D(l_T)$ is a non-positively curved squared complex. The virtual link l_T is prime if there are no reducing circles.

We do not change the genus of the projection surface F when triple l. The new links are added to the existing projection surface. The original l may have some reducing circles. The triple l, any previously existing reducing circle will become a closed curve that cuts the virtual link diagram in six places. A tripled link cannot form any additional reducing circles since every strand runs in sets of three between each square mesh of nine crossings.

CONCLUSION

It is still an open question whether the Writringer space of every virtual long knot – more generally, every labeled oriented tree – is spherical.

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